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A Note on Recurrent Random Walks on Graphs

András Telcs[1]

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We consider random walks on polynomially growing graphs for which the resistances are also polynomially growing. In this setting we can show the same relation that was found earlier but that needed more complex conditions. The diffusion speed is determined by the geometric and resistance properties of the graph.

KEY WORDS: Random walk; fractals; spectral dimension; potential theory.

1. FOREWORD

In our previous papers[6–8] we presented some results on diffusion properties of random walks (RW) on "*smooth*" graphs. It was clear that the smoothness condition is not very appealing, since it is hard to check for a given graph (although we presented some examples and pointed out our hope that most of the nonrandom fractal graphs have this property). This motivated us to present a more transparent set of conditions. The reader shall see that our condition on polynomially growing resistances can easily be verified and automatically holds for some classes of graphs. On the other hand, we think that this new approach may be useful for understanding the strength of our earlier "smoothness" condition better.[6]

2. INTRODUCTION

Let us consider an infinite, locally finite connected graph $G=(V, E)$ without loops. We use the convenient graph distance $d(x, y)$, the length of the shortest path between $x, y \in V$. The RW $(X_n)_{n \in \mathcal{N}}$ on V is a Markov chain with transition probabilities

$$P(X_{n+1}=y \mid X_n=x)=P(x, y)=\frac{1}{d_x}$$

[1] Library of Hungarian Academy of Sciences, 1361 Budapest, POB 7, Hungary.

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where $0 < d_x \leqslant D < \infty$ is the number of vertices adjacent to x. In other words, $(X_n)_{n \in \mathcal{N}}$ is a nearest neighbor RW on G. We want to study the mean exit time $T_{0,N}$ from the ball $B_{0,N-1}$ centered in 0 with radius $N-1$. We use the notation $S_{0,N} = B_{0,N} \setminus B_{0,N-1}$ for the surfaces of the balls. We apply the useful electric network model of G placing unit resistors at the edges. In this sense we can speak about the resistance $R(a, b)$ between two vertices $a, b \in V$ or resistance $R(A, B)$ between two sets $A, B \subset V$. Here the resistance between subsets means that we short-circuit the sets, i.e., they shrunk into a single point. Let us use the notation $a_n \asymp b_n$ to denote that $\log(a_n)/\log(b_n) \to 1$ as $n \to \infty$. Basically, we want to study the fractal dimension $d(0)$, the resistance dimension $d_\Omega(0)$, and the random walk dimension $d_R(0)$,

$$|B_{0,N}| \asymp N^{d(0)} \tag{1}$$

$$R(0, S_{0,N}) \asymp N^{2-d_\Omega(0)} \tag{2}$$

$$E(T_{0,N}) \asymp N^{d_R(0)} \tag{3}$$

where $0 \in V$ is a fixed vertex, the starting point of the RW $(X_0 = 0)$. Let us remark that an RW is recurrent if and only if the resistances $R(0, S_{0,N})$ tend to infinity as $N \to \infty$.[3] In this sense it is natural to say that a RW is *strongly recurrent* if $R(0, S_{0,N})$ is polynomially growing, i.e., $d_\Omega(0) < 2$.

Using this terminology, the main result of this paper is as follows.

Proposition 1. If an RW is strongly recurrent on a locally finite graph (i.e., $0 < d_x \leqslant D < \infty$) and

$$R(0, x) \asymp d(0, x)^{2-d_\Omega} \qquad (x \in V)$$

then

$$d_R(0) = d(0) + 2 - d_\Omega(0)$$

For more formal definitions and other details we refer to ref. 8. In the rest of this paper we give the proof of our result, some comments, and some examples and counterexamples; finally, we make some remarks on the subject, including a general conjecture.

3. DISCUSSION

3.1. The Electric Network Model

One can consider the finite graph $B_{0,N}$ as an electric network.[2,3] The edges are unit resistors and we apply a voltage between the poles $0 \in V$ and

$S_{0,N}$ such that the current $i_N(0)$ flowing from 0 to $S_{0,N}$ along the network is unity. Here we compress the vertices of $S_{0,N}$ into a single one. Let us denote the effective resistance[2] between 0 and $S_{0,N}$ by

$$R_N = R_{0,N} = R(0, S_{0,N})$$

Let us consider the voltage v_y, $y \in B_{0,N}$, generated by the unit current. We put

$$v_0 = R_N$$

and

$$v_y = 0$$

for all $y \in S_N$. One can prove that (cf. ref. 2, p. 52) for $y \in B_{0,N}$

$$P(X_i \text{ reaches } 0 \text{ before } S_{0,N} \mid X_0 = y) = \frac{v_y}{R_N} \tag{4}$$

and if u_y is the expected number of visits of X_i to y before it reaches $S_{0,N}$ (assuming $X_0 = 0$), then for $y \in B_{0,N}$ (cf. ref. 2, p. 50)

$$u_y = d_y v_y \tag{5}$$

It is clear that

$$E(T_{0,N}) = \sum_{y \in B_{0,N}} u_y = \sum_{y \in B_{0,N}} d_y v_y \tag{6}$$

Let us remark that we can extend the potential function on the vertex set to the edges by considering edges as linear resistors with unit length so that the potential varies linearly between the ends. We can extend the vertex set to the resistor network by putting $W = E \times [0, 1]$ so that we introduce an arbitrary orientation of the edges and we identify the "zero end" to the tail, the "1 end" to the head. The point $w = (e, t) \in W$ divides the edge from x to y into two pieces of length t, $1 - t$, i.e., $w(e, 0) = x$ and $w(e, 1) = y$. The point w has

$$v_w = (1 - t)\, v_x + t v_y$$

potential, by definition. Using this convention (for a more formal definition see ref. 6) we can define equipotential surfaces and regions surrounded by them. The equipotential surface of potential level p is

$$\Gamma_p = \Gamma_{0,p} = \{w \in W \mid v_w = p\}$$

Proof of Proposition 1. Let us use the notation $T_N = T_{0,N}$. The upper estimate for $E(T_N)$ is immediate by (6):

$$E(T_N) = \sum_{x \in B_{0,N}} u_x \leqslant D \sum_{x \in B_{0,N}} v_x \leqslant |B_{0,N}| \, R(0, S_{0,N})$$

Let us observe that this step does not need the conditions.

Now we start the lower estimate with some definitions. For brevity we use $R_N = R(0, S_{0,N})$. Let us define

$$s = s_N = \min\{d(0, y) \colon v_y < \tfrac{1}{2} R_N\} - 1$$

and choose y so that $d(0, y) = s + 1$ and $v_y < \frac{1}{2} R_N$. By the definition of s

$$v_x \geqslant \tfrac{1}{2} R_N$$

for all $x \in B_{0,s}$. Let us consider the potential surface Γ_{v_y} containing y. From the "shorting method"[6] we know that

$$R(0, y) \geqslant R(0, \Gamma_{v_y}) = R_N - R(\Gamma_{v_y}, S_{0,N})$$

But $R(\Gamma_{v_y}, S_{0,N}) = v_y$ and consequently

$$R(0, y) \geqslant \tfrac{1}{2} R_N$$

On the other hand, our condition ensures that

$$R(0, y) \asymp (s+1)^{2 - d_\Omega}$$

from which it follows that

$$s \asymp N$$

Consequently the number of vertices in the ball is given by

$$|B_{0,s}| \asymp N^{d(0)}$$

which results in

$$E(T_N) = \sum_{x \in B_{0,N}} u_x \geqslant \sum_{x \in B_{0,N}} v_x \geqslant \sum_{x \in B_{0,s}} v_x \geqslant \tfrac{1}{2} R_N \, |B_{0,s}|$$

and hence the statement. ∎

Using Kesten's terminology we can give a slightly stronger result. Let us define $\tilde{B}_{0,N}$ as the "backbone" of $B_{0,N}$, i.e., it consists of vertices from $B_{0,N}$ having disjoint path to 0 and to $S_{0,N}$.

Proposition 2. If an RW is strongly recurrent and

$$R(0, x) \asymp d(0, x)^{2 - d_\Omega(0)}$$

for $x \in \tilde{B}_{0,N}$, then

$$d_R(0) = d(0) + 2 - d_\Omega(0)$$

Proof of Proposition 2. It is enough to see that y chosen during the previous proof should belong to the backbone and the remaining part of the proof is working. Let us suppose that y does not belong to the backbone and not connected directly to 0 or to $S_{0,N}$. From the definition of the backbone it follows that there is a $z \in B_{0,N}$ such that if we remove z, then y is disconnected from 0 and $S_{0,N}$. But it means that $v_z = v_y$ and $d(0, z) < d(0, y)$, which contradicts the choice of y. ∎

Remark 1. From the proofs it is clear that if $d_\Omega = 2$ but some circumstances ensure that from

$$R(0, y) \geqslant \tfrac{1}{2} R_N$$

it follows that

$$d(0, y) \asymp N$$

then we can get the same conclusion as in our proof.

Remark 2. In ref. 8 we studied the corresponding Laplace operator with absorbing boundary (cf. also ref. 4). The submatrix $Q_{0,N}$ of P on $B_{0,N}$ is substochastic

$$Q_{0,N}(y, z) = P(y, z)$$

The Laplace operator is

$$\Delta_{0,N} = D_{0,N} - D_{0,N} Q_{0,N}$$

where for $y, z \in B_{0,N}$

$$D_{0,N} = (d_{y,z})_{y, z \in B_{0,N}}$$

and $d_{y,z} = d_y$ if $y \doteq z$, and 0 otherwise. The smallest eigenvalue of $\Delta_{0,N}$ will be denoted by $\mu_N(0)$ and its exponent by $d_\mu(0)$,

$$\mu_{0,N} \asymp N^{-d_\mu(0)}$$

In ref. 8 we defined another set of exponents e, e_Ω, e_R, and e_μ corresponding to the d's by

$$e_a = \limsup_N \sup_{y \in V} \frac{\log(a_N(y))}{\log(N)} \tag{7}$$

where $a_N(y) = b_{y,N}$, $R_{y,N}$, $E_y(T_N)$, or $\mu_N(y)^{-1}$, respectively (see details in ref. 8). We proved in ref. 8 (Theorem 2) that for recurrent RWs

$$e_R - \varepsilon \leqslant e_\mu \leqslant e_R$$

where $\varepsilon = e + 2 - e_\Omega - e_R$. This means that under the condition of Proposition 1 or 2 we get that $\varepsilon = 0$ and hence

$$e_\mu = e_R = e + 2 - e_\Omega \tag{8}$$

Remark 3. Some well-known graphs satisfy our conditions, namely Z^d for $d \leqslant 2$ (cf. Remark 1) and the Sierpinski gasket[5] embedded in the $\tilde{d}$-dimensional Euclidean space with dimensions

$$d = \frac{\log(\tilde{d}+1)}{\log(2)}$$

$$d_\Omega(0) = 2 - \frac{\log(\tilde{d}+3) - \log(\tilde{d}+1)}{\log(2)}$$

Remark 4. It is clear that our condition on $R(0, x)$ for "treelike" $d_\Omega > 1$ graphs does not hold, but for such graphs it is hoped that other methods can be found to study them.

Remark 5. Unfortunately, our method does not work in the case $d_\Omega(0) = 2$ in general, since in this case R_N can increase very slowly (e.g., $\log\log N$) and we have no guess on how to control the distances on a potential surface. We want to mention that we believe that

$$d_R(0) = d(0) + 2 - d_\Omega(0)$$

holds under quite weak assumptions for recurrent and transient RWs as well.

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